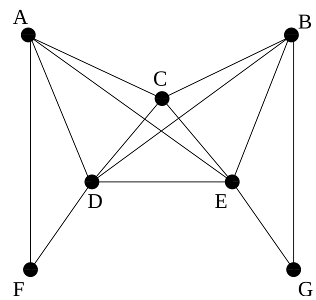
MATH 103: Contemporary Mathematics

Study Guide for Chapter 2: Hamilton Circuits

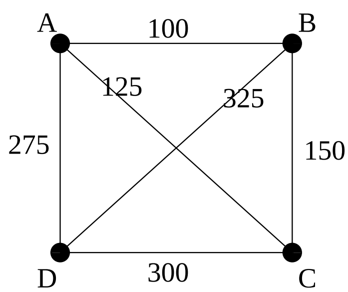
Note: We covered Sections 2.1 and 2.2 through page 48.

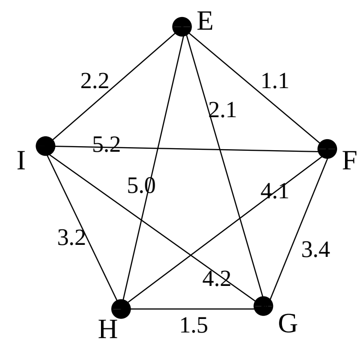
1. Answer the questions above each of the following graphs:

   (a) Find 3 different Hamilton circuits for the graph below.

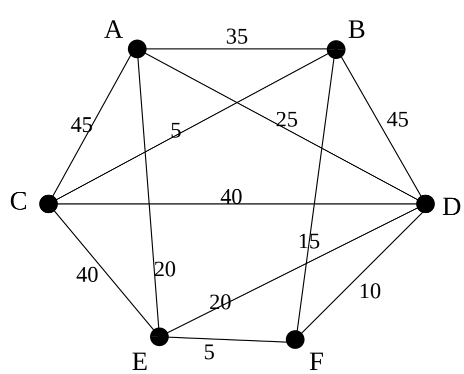


   (b) Practice a) the *Brute Force Algorithm*, b) the *Nearest Neighbor Algorithm*, and d) the *Sorted Edge* Algorithm (this is also called *Best Edge* or *Cheapest Edge Algorithm*) for the two graphs below:





   (c) Find the solution of the TSP problem for the following graph. (Careful! the graph is not complete. Also, we want you to find the solution, NOT an approximate solution.)



2. Explain the differencea between an Euler circuit and a Hamilton circuit.

3. Is there a rule like Euler's Theorem for Euler circuits which tells you when a graph has a Hamilton circuit?

4. (a) Within a Hamilton circuit (considered as a graph on its own), every vertex has valence ________

   (b) If a graph has a vertex of valence one, does this graph contain a Hamilton circuit? Explain.

   (c) If a graph has a vertex of valence one, can this graph contain a circuit? Explain.

   (d) If a graph contains a Hamilton circuit, then every vertex of the graph has valence at least ________

   (e) If every vertex of a connected graph has valence two or more, does the graph necessarily have a Hamilton circuit? Explain. If you think the answer is "no", provide an example.

5. If you are having trouble remembering the difference between an Euler circuit and a Hamilton circuit, you may want to recall some of the actual applications in each case. For example, police patrol, Chinese postman problem and garbage collection routes are modelled by Euler Circuits. Delivery problems, such as the one of a travelling salesman, are modelled by Hamilton Circuits. Explain why these two types of problems require different graph theory models.

6. Practice finding Hamilton circuits and computing their total weight by working problems 6,7,11,14 on pages 115-116.

7. In the following exercises the words **complete** and **connected**, are crucial. Review their definitions before answering the following questions.

   (a) What is a connected graph?

   (b) What is a complete graph?

   (c) Is any connected graph necessarily complete? Is any complete graph necessarily connected. Explain.

   (d) A graph is connected if any two vertices are ______________.

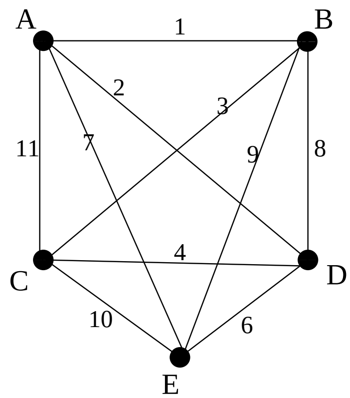
   (e) A graph is complete if any two vertices are ______________ .

8. A complete graph with three or more vertices always has a Hamilton circuit. How do we know this?

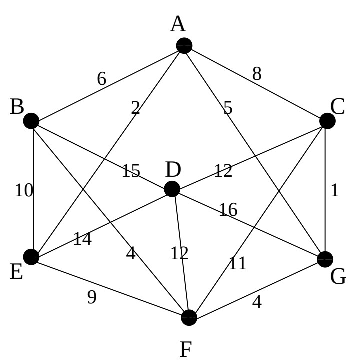
9. In a complete graph with N vertices there are _______________ edges. Review our class notes and problem 33p. 68 carefully before answering this question.

10. In a complete graph with N vertices there are _______________ Hamilton circuits starting at a given vertex, and _______________ distinct Hamilton circuits starting at a given vertex. Carefully explain how we derived each of the formulas. Review the *Fundamental Principle of Counting* on p. 42-43 before doing so.

11. Practice the Nearest Neighbor Algorithm (review it at page 45) for the graph below. Use A as your starting vertex. Indicate your answer as a list of vertices and compute its total weight. Then practice the Sorted Edge Algorithm: is the solution provided by it a better one?

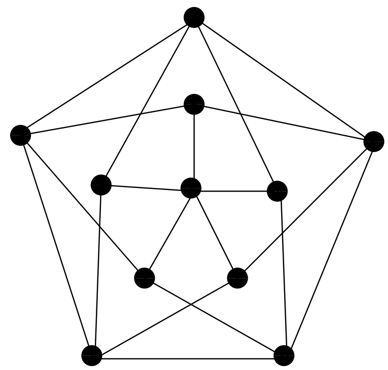


12. Use the Sorted Edge Algorithm (review it at page 46) to find a Hamilton circuit for the following weighted graph. Show your work (you can list the edges in the order you choose them). At the end describe your circuit as a list of vertices in the order in which you "visit" them. Are you allowed to visit any vertex more than once? Compute the weight of the circuit.



13. Provide an example of a graph which has a Hamilton circuit (come up with your own example, do not copy it from the book or the notes). Highlight the Hamilton circuit in your graph.

14. Use any method to find a Hamilton circuit in the following graph. Highlight your Hamilton circuit on the graph.



15. Which algorithm(s) of Chapter 2 (Brute Force, Sorted Edge, Nearest Neighbor) produce optimal (minimal weight) Hamilton circuits? Which one(s) produce Hamilton circuits which are (in most cases) only close to optimal?

16. Read about the Brute Force algorithm and the section at pages 101-106 to appreciate why the Brute Force algorithm is not an efficient method of solving the Travelling Salesman Problem with 20 or more cities (vertices).

17. If a graph has a Hamilton circuit, then the Brute Force algorithm will always yield an optimal Hamilton circuit. Why?

18. In the Nearest Neighbor Algorithm, are we allowed to visit a vertex (other than the starting vertex) twice? Why or why not?

19. In the Sorted Edge algorithm, we discard an edge if marking it would lead to three marked edges out of a single vertex. Why?

20. (More difficult question:) In the Sorted Edge algorithm, we discard an edge if marking it would lead to a marked circuit which does not include all vertices. Why? (Careful: the sentence *we want a circuit which includes all vertices* is not a complete answer).

21. Draw a connected graph with six vertices and eighteen edges. Does it have a Hamilton circuit? If not, then add a few edges so it does. Next, randomly number its edges starting from 1. Now, try using the Nearest Neighbor algorithm to find a Hamilton circuit in your weighted graph.

    (a) Did the algorithm work?

    (b) Is the graph you have drawn complete?

    (c) Can you have a Hamilton circuit in a graph which is not complete?

22. Think about applying the Nearest Neighbor or the Sorted Edge algorithms to a connected weighted graph, focusing on the last edge which you mark. Both algorithms will yield a Hamilton circuit for every **complete** graph.

    (a) Will these algorithms (Nearest Neighbor or Sorted Edge) yield a Hamilton circuit for every connected weighted graph? Why or why not?

    (b) Will these algorithms yield a Hamilton circuit for every connected weighted graph which has a Hamilton circuit? In other words, suppose someone constructs a weighted graph which has a Hamilton circuit. Will the two algorithms be guaranteed to find Hamilton(s) circuits in this graph? Explain.

    (c) Will the Sorted Edge algorithm yield a Hamilton circuit for some connected graphs which are not complete? Why or why not?

    (d) Why does the Sorted Edge (or Nearest Neighbor) yield a Hamilton circuit for every complete weighted graph with three or more vertices? (note your answer to part (b) and be careful; it is not enough to know that a graph has a Hamilton circuit).

23. Can a circuit contain another smaller subcircuit? Can a Hamilton circuit contain a smaller subcircuit?

24. Consider a graph with N vertices. Explain why the following statement is true: *If this graph has a Hamilton circuit, then such a circuit must contain N edges of the graph.*

25. For the following two problems, review number 45 p. 71, and review the bonus problem assigned in class.

26. Construct an example of a complete weighted graph for which the Sorted Edge algorithm produces the worst possible Hamilton circuit (i.e. the one with the highest total weight). Careful: you will need to convince me that it is the worst one by listing all other hamilton circuits and computing their total weights.

27. Construct an example of a complete weighted graph different from the one above, for which the Nearest Neighbor Algorithm produces the worst possible Hamilton circuit.

28. The following table shows the costs (in dollars) of airplane tickets between any two of five different cities. Sophia, who has friends in four cities other than her hometown Albuquerque, wants to find the cheapest way to visit each of her friends during a single trip.

	Albuquerque	Boston	Cincinnati	Denver	Edmonton
Albuquerque		244	430	127	654
Boston	244		197	250	370
Cincinnati	430	197		350	222
Denver	127	250	350		470
Edmonton	654	370	222	470	

(a) Draw a weighted graph model.

(b) Use the Nearest Neighbor Algorithm to find an approximate solution of this TSP.

(c) Use the Sorted Edge Algorithm to find an approximate solution of the TSP.

(d) How many different Hamilton circuits would you need to check to guarantee that you have found the best route for Sophia?