DISTANCE, AREA, LIMITS OF SUMS AND THE DEFINITE INTEGRAL

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One of the beauties of mathematics is that often problems that seem to be very different turn out to have very similar mathematical representations and solutions. Two such problems are finding the distance traveled when we know a variable velocity, and finding the area of a region with curved boundary.

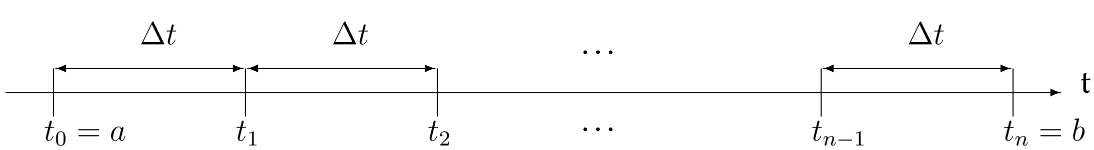
Distance from velocity

If we know a function that gives the velocity of an object at time t, that is we know $v = f(t)$ and we want to find the total distance s that the object travels over a time interval $a \leq t \leq b$, we can proceed as follows: Divide the time interval $[a, b]$ into n small subintervals each with length

$$\Delta t = \frac{b-a}{n}$$

and let $t_0, t_1, t_2, \ldots, t_n$ be the end points of the subintervals, so that

$$t_0 = a, t_i = a + i\Delta t, t_n = b$$



Over the i-th subinterval $t_{i-1} < t < t_i$, the velocity will be approximately a constant rate $v_i = f(t_i)$, so that we can use the familiar formula "distance = rate × time" to compute the distance s_i traveled (approximately) over that short time interval:

(1) $$s_i \simeq v_i \Delta t = f(t_i)\Delta t$$

Adding these up we get the total distance traveled is given (approximately) by

(2) $$s \simeq \sum_{i=1}^{n} f(t_i)\Delta t$$

If we used more subintervals (larger n and thus smaller Δt), we could get a better approximation, because the velocity would be closer to being constant over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta t \to 0$ of these approximating sums, then we can find the distance exactly:

$$s = \lim_{\Delta t \to 0} \sum_{i=1}^{n} f(t_i)\Delta t, = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)\Delta t$$

EVALUATING WITH CALCULATORS

An approximating sum such as (1) above may be easily calculated using a TI-86 calculator with the command

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sum seq(f(x)*h,x,a+h,b,h)
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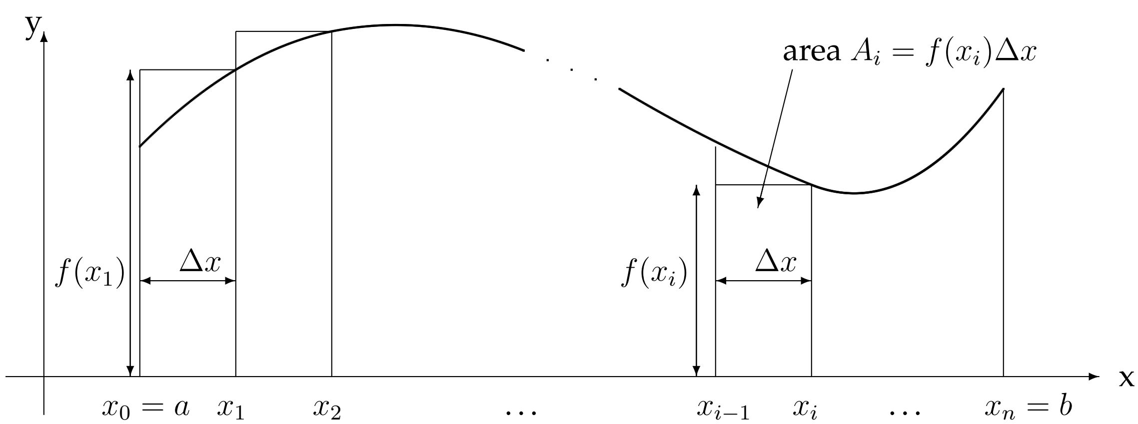
where

f(x)	is the velocity function (We use x instead of t because it is easier to enter.)
h	is Δt, the step size
x	is the variable being incremented: $x_1 = a + h$, $x_2 = a + 2h$, etc.
a+h	is the starting value for x, because the sum in (1) starts with $t_1 = x_1 = a + h$ instead of with $t_0 = a$.
b	is the ending value for x.

THE AREA OF A REGION WITH CURVED UPPER BOUNDARY

We can compute the area shown in the figure where the upper boundary is the curve $y = f(x)$, the lower boundary is the x axis, the left boundary is the line $x = a$, and the right boundary is the line $x = b$ as follows:



Divide the interval $[a, b]$ into n small subintervals each with length

$$\Delta x = \frac{b - a}{n}$$

and let $x_0, x_1, x_2, \dots, x_n$ be the end points of the subintervals, so that

$$x_0 = a, x_i = a + i\Delta x, x_n = b$$

The area A_i above the i-th subinterval $x_{i-1} < x < x_i$ will be approximately the area of a rectangle with width Δx and height $f(x_i)$:

$$A_i \simeq f(x_i)\Delta x$$

Adding these up we get the total area is given (approximately) by

$$A \simeq \sum_{i=1}^{n} f(x_i)\Delta x \tag{3}$$

THE EXACT AREA USING LIMITS

If we used more subintervals (larger n and thus smaller Δx), we could get a better approximation, because the rectangles would fit the true area closer over the shorter intervals. If we can find the limit as $n \to \infty$ and $\Delta x \to 0$ of these approximating sums, then we can find the area exactly:

$$A = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{4}$$

DEFINITION OF THE DEFINITE INTEGRAL

There are many other problems that can be calculated by this process of approximating with a sum of function values times a small interval width Δx and then finding the limit as the number of function values goes to ∞ and Δx goes to 0. Thus we need a name and symbol for it:

Definition. If $f(x)$ is a continuous function on the interval $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\Delta x, = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x \tag{5}$$

where $\Delta x = \dfrac{b-a}{n}$ and $x_i = a + i\Delta x$, is called the **definite integral** of $f(x)$ over the interval $[a, b]$.

Note that the notation modifies the "Greek S" Σ to become the "elongated S" $\int$ and changes the "Greek D" Δ in Δx to the "d" in dx, to indicate that the limit was taken as $n \to \infty$ and $\Delta x \to 0$.

OTHER CHOICES OF INTERVAL HEIGHTS AND WIDTHS, LIKE THE MID-POINT RULE

The way we have set up the approximating sums above always used rectangle heights computed at the right end point of each subinterval, because that makes the notation easiest. The text (Stewart, section 5.1) points out that the heights can instead be computed at other points x_i^* in each interval, like the left endpoint or the midpoint.

Of these options, using the mid-point of each interval is intuitively the best choice, and this in fact can be proven to be the most accurate in some sense, to be seen in Calculus 2.

Also, the subintervals do not all have to be the same width, as long as when the number of subintervals $n \to \infty$, the width of the widest subinterval $\to 0$. The limit of all these different approximating sums as $n \to \infty$ will be the same if $f(x)$ is a continuous function, but that fact, or even that the limit exists, is a fairly difficult theorem to prove in advanced calculus. Read the text to get some idea of why it is true.

Homework Assignment on Definite Integrals

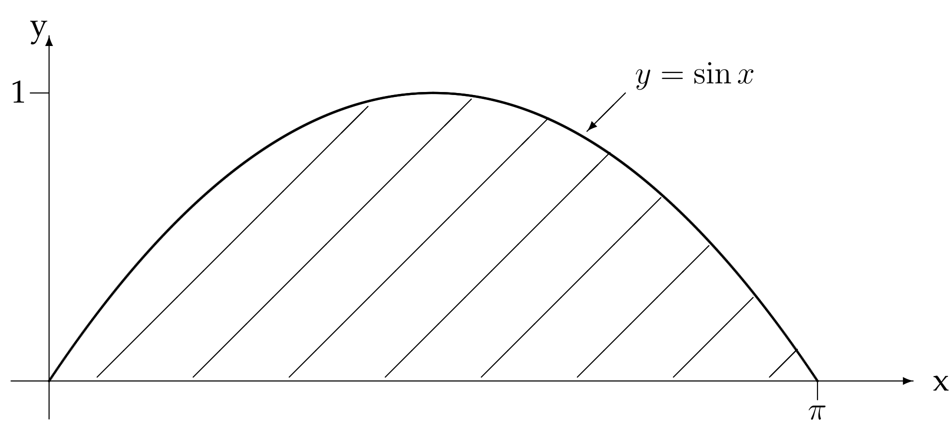
(1) Write the following limits of sums as definite integrals:
   (a) $\lim_{\Delta x \to 0} \sum_{i=1}^{n} (x_i^2 - x_i)\Delta x$, where $\Delta x = \frac{5-1}{n}$, $x_i = 1 + i\Delta x$
   (b) $\lim_{\Delta x \to 0} \sum_{i=1}^{n} (\sin x_i)\Delta x$, where $\Delta x = \frac{\pi - 0}{n}$, $x_i = 0 + i\Delta x$

(2) Write the following definite integrals as the limit of a sum:
   (a) $\int_3^7 \sqrt{49 - x^2}\, dx$
   (b) $\int_0^8 2 + x^2\, dx$

(3) For each definite integral in problem 2, do the following:
   (a) Draw an area which is equal to the definite integral.
   (b) Draw four rectangles whose area approximates the area in part (a).
   (c) Label the points x_1, x_2, x_3, x_4 on your drawing and indicate line segments with length Δx and $f(x_1)$.

(4) The velocity of a UFO is given by $v = t + t^2$ miles/sec for the time interval $1 \le t \le 4$.
   (a) Write approximating sums for the total distance traveled over the interval by the UFO using $n = 4, 20$ and 100 subintervals.
   (b) Use your calculator to evaluate each of the sums in part (a).
   (c) Write an expression for the exact distance using the limit of a sum.
   (d) Write an expression for the exact distance using the definite integral.
   (e) Based on your approximations in part (b), guess the exact value of the distance; i.e. guess the value of the limit.

(5) For the region shaded in the figure below



   (a) Write approximating sums for the area using $n = 4, 20$ and 100 subintervals.
   (b) Use your calculator to evaluate each of the sums in part (a).
   (c) Write an expression for the exact area using the limit of a sum.
   (d) Write an expression for the exact area using the definite integral.
   (e) Based on your approximations in part (b), guess the exact value of the area; i.e. guess the value of the limit.