# MATH 220 MOCK FINAL EXAM BRIEF ANSWERS

(1) (a) $\lim_{x\to\infty} e^{-x}\ln x = 0$

(b) $y = (1+x^2)\tan^{-1}x$, $dy/dx = 1 + 2x\tan^{-1}x$

(c) $f(x) = \tanh(\ln x)$, $f'(x) = \frac{\operatorname{sech}^2(\ln x)}{x}$

(d) $\displaystyle\int \frac{3x+2}{x^2+6x+8}\,dx = 5\ln|x+4| - 2\ln|x+2| + C$

(e) $\displaystyle\int x\sin 4x\,dx = \frac{\sin 4x}{16} - \frac{x\cos 4x}{4} + C$

(f) $\displaystyle\int \sin^2 x\cos^3 x\,dx$

(g) $\displaystyle\int \frac{dx}{(4-x^2)^{3/2}} = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$

(2) $\displaystyle\int \tan^4(2x)\,dx = \frac{\tan^3 2x}{6} - \frac{\tan 2x}{2} + x + C$

(3) Write an expression in $\sum$ notation for the approximation of $\int_1^3 \sqrt{\sin x}\,dx$ is approximated by

$$\frac{1}{2}\sum_{i=1}^{4}\sqrt{\sin\left(\frac{3}{4}+\frac{i}{2}\right)}, = 1.546\ldots$$

(4) (a) $\displaystyle\int_{-8}^{27}\frac{dx}{x^{1/3}}$ is an improper integral because the integrand is undefined at the point $x = 0$ within the range of integration. It must be evaluated as the sum of two improper integrals

$$\int_{-8}^{0}\frac{dx}{x^{1/3}} + \int_{0}^{27}\frac{dx}{x^{1/3}}$$

The indefinite integral in each case is $\frac{3}{2}x^{2/3}$, so the limits needed converge, making it convergent with value

$$\frac{3}{2}(27^{2/3} - (-8)^{2/3}) = 15/2.$$

(b) $\displaystyle\int_2^\infty e^{-2x}\,dx = e^{-4}/2.$

(5) The partial fractions form is

$$\frac{x^2+2x+4}{(x^2+1)^2(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D+Ex}{x^2+1} + \frac{F+Gx}{(x^2+1)^2}$$

(6) The area of the region bounded by the curves $y = 2x^2$ and $y = x^3$ is $4/3$.

(7) (a) When the region in the first quadrant bounded by the $y$-axis, the curve $y = \sqrt{x}$, and the line $y = 4$ is rotated about the $x$-axis, the resulting solid has volume $128\pi$.

(b) The volume of the region produced by rotating this region about the $y$-axis is $1024\pi/5$.

(8) If the velocity of a car moving away from a stop sign $t$ seconds after starting to move is $v = 5t - t^2$ meters/second, its average velocity over the first two seconds is 22/6 m/s.

(9) The general solution of the differential equation

$$\frac{dy}{dx} = x^2 e^y$$

is $y = -\ln(C - x^3/3)$ and the particular solution that passes through the point $(4, 2)$ has $C = 8/3 + 1/e^2$, so is $y = -\ln(8/3 + 1/e^2 - x^3/3)$.

(10) The ant population will reach one billion in about $18\dfrac{\ln(100000/3)}{\ln(30)} \approx 55$ months from now.

(11) The linear differential equation

$$x^2\frac{dy}{dx} = \cos x - 2xy$$

has general solution $y = \dfrac{C + \sin x}{x^2}$.

(12) The parametric curve $x = \cos^2 t$, $y = \cos t$, $0 \le t \le 2\pi$ moves along the sideways parabola $x = y^2$ from $(1, 1)$ to $(1, -1)$ and back. It has slope $\dfrac{dy}{dx} = \dfrac{1}{2\cos t}$, so has vertical tangent when $t = \pi/2, 3\pi/2$. These two values are the two times that the curve passes through $(0, 0)$.

(13) The length of the parametric curve

$$x = e^{2t} - 2t, \quad y = 4e^t, \quad 0 \le t \le 2,$$

is

$$\int_0^2 \sqrt{(2e^{2t} - 2)^2 + (4e^t)^2} dt = \int_0^2 (2e^{2t} + 2)\, dt.$$

(14) The curve $r = \sqrt{\theta}$ in the $x$-$y$ plane for $0 \le \theta \le 2\pi$ is a spiral growing anti-clockwise going from the origin to $(\sqrt{2\pi}, 0)$.

(15) The area inside one loop of the polar curve $r = \sin 3\theta$ is $\displaystyle\int_0^{\pi/3} \frac{1}{2}\sin^2 3\theta\, d\theta = \pi/12$.

(16) A direct formula "$a_n$" and a recursive description for a sequence that starts

$$1, -\frac{2}{3}, \frac{6}{9}, -\frac{24}{27}, \frac{120}{81} \cdots$$

is $a_n = \dfrac{(-1)^{n-1} n!}{3^{n-1}}$, and a recursive description is $a_1 = 1$, $a_n = -n a_{n-1}/3$.

(17) (a) $\{a_n\} = \left\{\frac{n+\sin n}{2n+4}\right\} = \left\{\frac{1+\sin 1}{5}, \frac{2+\sin 2}{7}, \frac{3+\sin 3}{7}, \ldots\right\}$ and $\lim_{n\to\infty} \frac{n+\sin n}{2n+4} = 1/2.$

(b) $\{a_n\} = \left\{\frac{(-1)^n}{1+2/n}\right\} = \{-1/3, 1/2, -3/5, \ldots\}$ and the sequence has no limit because for large $n$, the values oscillates between near 1 and near $-1$.

(18) (a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} = -1 + 1/4 - 1/18 + 1/96 - \cdots$ The third and fourth partial sums are $S_3 = -28/36 = -7/9 = -0.777\ldots$, $S_4 = -221/288 = 0.76736111\ldots$

(b) This series converges due to the AST: the terms have alternating sign, decreasing magnitude, and the terms (not their sum!)converge to zero.

The partial sum is within 0.01 of the infinite sum once the first term not summed is smaller than 0.01. The next term not in $S_4$ is $-1/(5 \cdot 5!) = -1/600$, small enough, so summing the first four terms is enough.

(19) The quadratic Taylor polynomial approximation with center $a = 0$ for $f(x) = \sqrt[3]{x+8}$ is

$$T_2(x) = 2 + \frac{1}{12}x - \frac{1}{144}x^2$$

Thus $\sqrt[3]{9} \approx T_2(1) = 2 + \frac{1}{12} - \frac{1}{144} = 2.0763888\ldots$

(20) (a) $\sum_{n=0}^{\infty} \frac{(1+n^3)x^n}{n!} = 1 + 2x^2 + 9/2x^2 + \cdots$

(b) The radius of convergence of the series ins infinite: it converges for all $x$.

(21) No elementary anti-derivative exists for the function $\sin(x^2)$ but we can approach its integral as follows.

(a) $\sin(x^2) = x^2 - x^6/3! + x^{10}/5! = \sum_{n=0}^{\infty}(-1)^n/n!x^{4n+2}$.

(b) $\int \sin(x^2)\, dx = C + x^3/3 - x^7/(7\dot{3}!) + x^{11}/(11\dot{5}!) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)n!}x^{4n+3}$.

(c) Th series for $\sin x$ converges for all $x$, so in particular this is true when any value of $x^2$ is inserted in place of $x$. Thus the series in (a) has infinite radius of convergence, and integrating a series iv a new series with the same radius of convergence, so the series in (b) also has infinite radius of convergence.