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SOLUTIONS OF THE FISHER EQUATION IN EXTERNAL ELECTRIC FIELD

BY

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ABSTRACT: The present paper presents analytic solution of the Fisher equation with a galvanotaxis flux and a logistic term source. We find that when the electrical potential is spatially periodic appear corresponding spatial regions of high and low population density. Moreover, if the electrical potential is a time and spatial periodic quantity then the travel waves take place in the system. This is a nonlinear effect specific to the dissipate systems and reflect competition between diffusion and autocatalitic effects.

1.INTRODUCTION

Let us consider a spatially distributed system formed by some individuals (ions, cells, bacterias, etc) sensitive to an external electric field. Such systems are usually far from thermodynamic equilibrium and therefor the evolution equation of the macroscopic (contracted) variables write as balance equation. The time dependence of the concentration is called the Fisher equation. This equation has the general form [1,3]

$$\frac{\partial n}{\partial t} = -\nabla \cdot \vec{J} + \sigma(n), \qquad (1)$$

where n is the concentration of the specie, $\vec{J}$ is conventional flow term and $\sigma(n)$ is the source term.

In addition to the well-known diffusion term proportional with the gradint od the concentation we consider the diffusion induced by an external electric field

$$\vec{J} = \vec{J}_{diff} + \vec{J}_{el} = -D\nabla n + nf(V)\nabla V, \qquad (2)$$

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where f(V) reflects the electrical sensitivity of the individuals to the external electrical field. Usually the specific form of this function is experimentally obtained.

As a major simplification let us consider that the diffusion coefficient is just a constant quantity. Using (1) and (2) the evolution equation for the concentration becomes

$$\frac{\partial n}{\partial t} = D\nabla^2 n - f(V)\nabla V \cdot \nabla n - nf(V)\nabla^2 V + \sigma(n), \tag{3}$$

and can be rewritten in a more useful form

$$\frac{\partial n}{\partial t} = D\nabla^2 n + f(V)\vec{E} \cdot \nabla n + nf(V)\nabla \cdot \vec{E} + \sigma(n), \tag{4}$$

where $\vec{E} = -\nabla V$ is the intensity of the electric field.

Another major simplification is the assumption that the system evolves in a one-dimensional space.

Our principal goal is to find traveling waves solutions of the equation (4) because we expect an electrotaxis movement

$$n(x,t) = n(x-ct), \tag{5}$$

where *c* is the velocity of the wave front. Substituting (5) into the equation (4) one obtains

$$Dn'' + (c + Ef(V))n' + nf(V)\frac{dE}{dx} + \sigma(n) = 0, \tag{6}$$

where primes denote the derivatives of the density by the quantity $z=x-ct$.

2. STEADY STATES SOLUTIONS AND THEIR STABILITY

The above equation (6) is equivalent with the following system of equations [1,2,3]

$$\begin{cases} u_1' = u_2, \\ u_2' = -\dfrac{1}{D}\left[(c + Ef)u_2 + u_1 f\dfrac{dE}{dx} + \sigma(u_1)\right], \end{cases} \tag{7}$$

where $u_1 = n$.

The steady states solutions of the system (7) are solutions of the following system of equations

$$\begin{cases} u_1 f \dfrac{dE}{dx} + \sigma(u_1) = 0, \\ u_2 = 0. \end{cases} \tag{8}$$

The corresponding eigenvalues equation is

$$D\lambda^2 + (c + Ef)\lambda + \left(\sigma' + f\frac{dE}{dx}\right) = 0 \tag{9}$$

where primes denote derivatives of the function.

The solution of the equation (9) determines the stability condition of the steady states (8). To be more specific we will refer in the following to the Fisher equation with logistic source term

$$\sigma(n) = \alpha n^2 + \beta n \tag{10}$$

where $\alpha < 0$ and $\beta > 0$ are constants. Substituting (8) into the equation (10) results the steady states solutions

$$\begin{cases} u_1 = 0; u_2 = 0, \\ u_1 = -\dfrac{1}{\alpha}\left(\beta + f\dfrac{dE}{dx}\right); u_2 = 0. \end{cases} \tag{11}$$

Using (11) and (9) the eigenvalues equation in the steady state ($u_1 = 0$, $u_2 = 0$) becomes

$$D\lambda^2 + (c + Ef)\lambda + \left(\beta + f\frac{dE}{dx}\right) = 0 \tag{12}$$

and in the second steady state (11b) is

$$D\lambda^2 + (c + Ef)\lambda - \left(\beta + f\frac{dE}{dx}\right) = 0 \tag{13}$$

Two important response function of the biological systems are known from physiology

$$\begin{aligned} f(V) &= \frac{f_o}{V}, \\ f(V) &= \frac{f_0 A}{(A+V)^2}, \qquad f_0, A > 0 \end{aligned} \tag{14}$$

where f_0 are positive defined constants, A is a scaling potential (ground state potential).

The first response function is the logarithmic law and the second one is the receptor law. We adopt here the hypothesis that these two response functions are still valuable for the electrical potential.

The support for such an assumption is the experimental facts that chemical and electrical response of the receptors have the same similarity class.

2.1. SPATIAL DISTRIBUTED SYSTEM IN ELECTROSTATIC FIELD

One of the most simple situation concern the spatial distribution of individuals under external constant potential

$$V = V_0 = \text{constant} \tag{15}$$

From (12) and (13) it can be seen that the eigenvalues for the steady state (11a) are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4D\beta}}{2D} \tag{16}$$

and for the steady state (11b) are

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 + 4D\beta}}{2D} \tag{17}$$

In order to have a traveling wave solution all the eigenvalues must be real quantities. From (16) then follows that

$$c^2 - 4D\beta \geq 0 \tag{18}$$

Therefore the steady state (11a) is a stable node, according to (16), and the steady state (11b) is an unstable saddle point. In these conditions there is an heterocline orbit that connects the two steady states and therefore exists a traveling wave solution of the problem (see Fig.1). Moreover, from (18) it is possible to find the minimum velocity of the wave front.

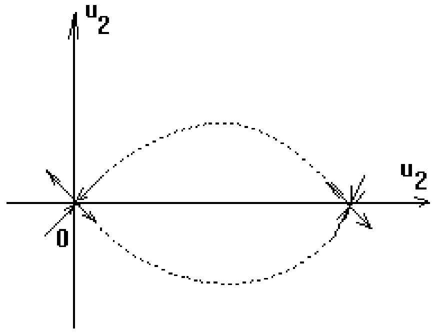

Fig.1 Phase portre of the Fisher equation.

2.2. SPATIAL DISTRIBUTED SYSTEM IN UNIFORM ELECTRIC FIELD

Let us consider that the electric potential has the particular form

$$V = a\,x + b \tag{19}$$

where *a, b* are positive defined constants. This case corresponds to constant electric field. Following the same technique as in the preceding subsection the corresponding eigenvalues are:

$$\lambda_{1,2} = \frac{-\left(c - \frac{aA}{ax+b}\right) \pm \sqrt{\left(c - \frac{aA}{ax+b}\right)^2 - 4D\beta}}{2D} \tag{20}$$

for the steady state (11a) and

$$\lambda_{1,2} = \frac{-\left(c - \frac{aA}{ax+b}\right) \pm \sqrt{\left(c - \frac{aA}{ax+b}\right)^2 + 4D\beta}}{2D} \tag{21}$$

for the steady state (11b). The eigenvalues (20) must be real quantities and therefore

$$c \ge 2\sqrt{D\beta} + \frac{aA}{ax+b}. \tag{22}$$

To obtain these results the logarithmic law was used. From (22) results that the velocity of the wave front is not constant. The maximum value corresponds to the source term position (it happens to be here the origin of the spatial frame). The wave front has a constant velocity when $x \to \pm\infty$.

3.CONCLUSIONS

The present study demonstrates that the excitable environement can develop a wide class of nonlinear phenomena. The most spectacular one is the traveling waves solutions for the spatially distributed systems. The existence of the minimum velocity of the wave front was experimentally reported previously. The main application of the results concerns the electric response of the excitable environement (cardiac tissues, neuronal network and other selforganized biological systems).

The Fisher model is one of the most simple one but even so it resembles features of real behavior of the complex systems and therefore can be the bases of more elaborate studies.

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